Lecture 5: Score and Fisher Information
Definition: Score function

\[ S(\theta) = \frac{\partial}{\partial \theta} \log L(\theta) \]

Consequently, the MLE is obtained from the score equation \( S(\theta) = 0 \)

Definition: Fisher Information

\[ I(\theta) = -\frac{\partial^2}{\partial \theta^2} \log L(\theta) \]

Standard errors are computed from the Fisher Information: \( se(\hat{\theta}) \equiv I^{-0.5}(\theta) \)
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Example 2.9

Let $x_1, x_2, \ldots, x_n$ be an iid sample from $N(\theta, \sigma^2)$. Assume $\sigma^2$ known.

\[
\log L(\theta) = -\frac{1}{2\sigma^2} \sum (x_i - \theta)^2
\]

\[
S(\theta) = \frac{\partial}{\partial \theta} \log L(\theta) = \frac{1}{\sigma^2} \sum (x_i - \theta)
\]

Score equation $S(\theta) = 0 \Rightarrow \hat{\theta} = \bar{x}$

Observed Fisher Information $I(\hat{\theta}) = -\frac{\partial^2}{\partial \theta^2} \log L(\hat{\theta}) = \frac{n}{\sigma^2}$

$\Rightarrow \text{se}(\hat{\theta}) = \sqrt{\frac{\sigma^2}{n}}$
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Multiparameter models: Example 3.7

Let \( x_1, x_2, \ldots, x_n \) be an iid sample from \( N(\mu, \sigma^2) \). Both \( \mu \) and \( \sigma^2 \) not known.

Score functions

\[
S_1(\mu, \sigma^2) = \frac{\partial}{\partial \mu} \log L(\mu, \sigma^2) = \frac{1}{\sigma^2} \sum (x_i - \mu)
\]

\[
S_2(\mu, \sigma^2) = \frac{\partial}{\partial \sigma^2} \log L(\mu, \sigma^2) = -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum (x_i - \mu)^2
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The score equations \( S_1(\mu, \sigma^2) = 0 \) and \( S_2(\mu, \sigma^2) = 0 \) gives the MLE

\[
\hat{\mu} = \bar{x}
\]

\[
\hat{\sigma}^2 = \frac{1}{n} \sum (x_i - \bar{x})^2
\]

The second derivatives give the observed Fisher Information matrix

\[
I(\mu, \sigma^2) = \begin{pmatrix}
\frac{n}{\sigma^2} & 0 \\
0 & \frac{n}{2(\sigma^2)^2}
\end{pmatrix}
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