A Comparison of Monte Carlo Simulation and Quasi-Monte Carlo Simulation for American Option Pricing

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Abstract

Monte Carlo Simulation is regarded as one of the most powerful approach in American-style option pricing. Some of its characters, such as the fast convergence rate, the small amount of computation and escape from the so-called “dimension curse”, enhance its ability of wide application. Meanwhile, the Quasi-Monte Carlo simulation which uses the low-discrepancy sequences instead of the pseudorandom numbers sequences as the fundamental samples of random numbers presents a more attractive result in European option pricing. Thus, the purpose of this thesis is to compare the performance of the MC and QMC in American option. First, we separately apply the MC and QMC method, which both based on Least-squares Monte Carlo algorithm, to an American option with a single underlying asset. Second, compare performances of these two methods based on computed option value, standard error and computation time. The results show that the Quasi-Monte Carlo simulation method is better than Monte Carlo method in terms of variance reduction.

Key words: American Option Pricing; B-A-W Method; Low-discrepancy Sequence
1. Introduction

As a kind of financial innovation, options are first developed in the United States in 1970s. It is reported by the Chicago Board Options Exchange\(^1\), which is the biggest option exchange market in the world, that until April, 2009 there are more than 70 different types of options in the transaction. Generally speaking, an option is a contract between a buyer and a seller that gives the buyer the right—but not the obligation—to buy or to sell a certain asset for a specified price (Brandimarte, 2006). We know that option price is the only variables, whose value always changes due to the uncertainty of supply and demand of market, in an option contract. Whether the price of an option is reasonable directly affects the buyer and seller's profit. Thus how to fairly value or price the financial product such as the options become a popular topic both in theoretical and practical field.

According to the difference of exercise date, options can be divided into two types, American options and European options. The American options have a feature that it allows the option holders to exercise it on any available date before the date of expiration, rather than to exercise it on a fixed date like the European options do. Due to the complexity in the contract, the pricing of an American option does not have an exact analytical solution. Analytical approximation method and numerical analysis methods are widely used to approximate the option price. The most famous approximate analytical method is first proposed by Macmillan (1986) and extended to quadratic approximation.

\(^1\) Chicago Board Options Exchange: http://www.cboe.com/data/
method suggested by Barone-Adesi and Whaley (1987), thus it is also known as BAW method. However, in recent years, numerical analysis method, including the binary tree method, finite difference method, as well as Monte Carlo (MC) simulation, become a commonly used pricing method than any other ones. Cox, Ross and Rubinstein (1979) first proposed the binary tree model to solve American option pricing; Brennan and Schwartz (1978) applied finite difference method to estimate American option price; Tilly (1993) first introduce the Monte Carlo simulations to the American option pricing. Though Monte Carlo is regarded as one of the most powerful approach, in some cases, Quasi-Monte Carlo (QMC) presents more attractive results than MC. The results above have been examined for the European options (Kind. A, 2005). Hence, the aim of this paper is to test whether QMC performs better in the case of the American options, too.

The structure of this paper is organized as follows: Section 1 is an introduction to some basic concepts of option and summarizes the relevant recent works. Section 2 gives a general view of American option pricing. The development of pricing method first started with European options, and some assumptions under European options are also consistent with American option. Therefore, in this section, we introduce the Black-Scholes model first and then give a review of BAW method and finally compare some numerical analysis methods. Section 3 focuses on the Monte Carlo simulation method, and Section 4 describes the Quasi-Monte Carlo simulation methods. Section 5 presents empirical experiments to compare the performance of the Monte Carlo simulation and the Quasi-Monte Carlo simulation for American option pricing.
Finally, section 6 presents summary conclusions. If not specified, the discussion of this article is based on stock options.

## 2. American Option Pricing

As mentioned above, the features of the early exercises of American options imply the higher benefit for the purchaser than the European option; consequently, the American option’s price is higher. Due to the flexibility of American option in terms of exercise date, the American option is the majority of universal option trade now (Tang, M., 2007).

The development of option pricing theory is basically associated with the enrichment of financial products. After absorbing the analysis and modeling techniques of physics, mathematics, and computer science and so on, the option pricing model has achieved a great improvement both in precision and execution efficiency. And, most of the researches run under Black-Scholes model (Ma, et al, 2000). It can be said that the Black-Scholes model established a firm base for some other new financial derivative pricing theories. In order to honor the big success of the research, the Sweden’s Central Bank awarded the Nobel Prize in Economics for 1997 to professor Myron Scholes of Stanford University and Robert C. Merton of Harvard University.

### 2.1 The Black-Scholes Model

Black-Scholes (B-S) model has five hypotheses:

1. Securities business is a weak efficient market.

2. All investors are in a risk neutral environment. All negotiable securities’ income rates are non-risk interest rate.
3. No trading fee, tax and the difference between selling and buying price.

4. People can put or get money in the non-risk interest rate at anytime.

5. Stock price follows Geometric Brownian motion which can be presented as follow:

\[ dS(t) = rS(t)dt + \sigma S(t)dW(t) \]

where, the continuous random variable \( S(t) \) is stock price at time \( t \), \( W(t) \) is a Wiener process and \( r \) and \( \sigma \) are drift and volatility respectively. But, in the case of finance, they are defined as the risk free rate and the standard deviation of the stock price.

According to Ito’s theory, we know that any derivatives based on non-dividend stock satisfy the following equation:

\[ dF = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial S} dS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} dt \]  

(2.1)

Where \( F \) is value of the option at a time \( t \). \( F \) is a function depending only on time and current price of the underlying asset.

In order to remove the random item \( dS \), we build a portfolio consisting of \( \Delta \) unit of the original share, \( S \), and one unit of the option written on it. The value of this portfolio is deterministic:

\[ \Pi = \Delta \cdot S - F(S,t) \]  

(2.2)

Where \( \Delta \) is a certain number of stock shares and \( \Delta \in \mathbb{R} \)

Differentiating equation (2.2) and substituting it into (2.1), we get:

\[ d\Pi = \Delta dS - dF = (\Delta \frac{\partial F}{\partial S})dS - \frac{\partial F}{\partial t} dt + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} dt \]  

(2.3)

So, let \( \Delta = \frac{\partial F}{\partial S} \), the combination \( \Pi \) doesn’t have the random item \( dS \). It
shows that after time \( dt \), the value of combination doesn’t have risks.

According to non-arbitrage balance theory, combination income rate should equal to non-risk negotiable securities’ income rate. So:

\[
d \Pi = r \Pi dt
\]

(2.4)

Where \( r \) is risk free rate.

Combination (2.3) and (2.4), we can get:

\[
\frac{\partial F}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} dt = r(F - S \frac{\partial F}{\partial S}) dt
\]

Rewrite the equation above, we get:

\[
\frac{\partial F}{\partial t} + rS \frac{\partial F}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} - rF = 0
\]

(2.5)

Equation (2.5) is the famous Black-Scholes differential equation. This equation could determine any option price whose payoff on exercise date only depends on the current price of underlying asset. Generally speaking, solutions to partial differential equation also depend on boundary condition and initial state. For example, European call option’s boundary condition is the price on the expiration day. \( F(S,T) = \max \{ S - K, 0 \} \), where \( K \) is exercise price, and then we can get the analytical solution of \( F(S,T) \):

\[
F(S,t) = S_0 N(d_1) - Ke^{-r(T-t)} N(d_2)
\]

Where:

\[
d_1 = \frac{\ln(S_0 / k) + \left(r + \frac{1}{2} \sigma^2\right)(T-t)}{\sigma \sqrt{T-t}}
\]

\[
d_2 = \frac{\ln(S_0 / k) + \left(r - \frac{1}{2} \sigma^2\right)(T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t}
\]
and $N(x)$ is the standard normal distribution function.

According to put-call parity relationship (Stoll, 1969), the analytical solution of European put option in the same condition is:

$$P(S, t) = Ke^{-r(T-t)}N(-d_2) - S_0N(-d_1)$$

From the derivation above, we can see that the biggest contribution of Black-Scholes model is to ensure that the partial differential equation of European option has a fixed analytical solution in non-arbitrage complete market hypothesis. The option price is related to stock price, time, variance of stock price and risk-free interest rate, but has no relation with risk favor of the holder.

2.2 Methods of American Option Pricing

The assumptions of European option pricing also serve American option pricing, such as stock price follows log-normal distribution and stock income rate follows geometric Brownian motion. Under these hypotheses, considering that American option can be exercised at any time during a valid period, the question transformed to compare internal value to holding-on value in the certain time $t (t<=T)$. If the former one is bigger, then exercise the option at once, else wait for the biggest benefit of time. This question is a freedom margin of partial differential equation in mathematic. But European option has a fixed marginal condition while American option hasn’t, so we can’t get the fixed analytical solution. We need to approximate it in other method. This is why American option pricing is difficult.

2.2.1 Quadratic Approximation Method

Barone-Adesi and Whaley (1987) introduced the analytical quadratic
approximation method (BAW). Unlike other numerical methods, the main idea of BAW method is to decompose the American value into European value and early exercise premium. For example, the early exercise premium of American call option $e(S, T)$ can be defined as:

$$e(S, T) = C(S, T) - c(S, T)$$

where $C(S, T)$ is the price of American call option and $c(S, T)$ is the price of European call option. If the price of American call option satisfies Merton’s partial differential equation, then the early exercise premium should satisfy it as well (Barone-Adesi and Whaley, 1987). That is

$$\frac{1}{2} \sigma^2 S^2 e_{ss} - re + bSe_s + e_t = 0$$

where, $\sigma$ is the standard deviation of underlying asset, $r$ is the risk free rate of interest and $b$ is the cost of carrying. For a non-dividend-paying stock, which is applied in this paper, the cost of carrying should be equal to risk free rate of interest ($b=r$). $e_s, e_t$ defines the partial derivative of $e(S, T)$ with respect to $s$ and $t$, respectively. $e_{ss}$ is the second order partial derivative with respect to $s$.

In order to simplify the calculation, we let

$$\tau = T - t$$  

$$H(\tau) = 1 - e^{-\tau r}$$  

$$M = 2r / \sigma^2$$  

$$N = 2b / \sigma^2$$

Then, (2.6) could be rewritten as:

$$S^2 e_{ss} - Me + NSe_s - (M / r)e_t = 0$$  

(2.7)

After the transformation, the early exercise premium could be expressed as

$$e(S, H) = H(\tau) f(S, H)$$

Substitute it to the (2.7), we get
\[ S^2 f_{ss} + N S f_s - \frac{M}{H(\tau)} f - (1 - H(\tau)) M f_H = 0 \]  
\[ (2.8) \]

When \( \tau \) is large, \( 1 - H \) goes to infinity; when \( \tau \) is small, \( f_H \) goes to infinity as well. Thus, we could neglect the last term of left side. (2.8) approximately equal to
\[ S^2 f_{ss} + N S f_s - \frac{M}{H(\tau)} f = 0 \]

Then, we could derive the analytical solution of American call option \( C(S, \tau) \)
\[
C(S, \tau) = \begin{cases} 
  c(S, \tau) + A_2 \left( \frac{S}{S^*} \right)^{q_2}, & \text{when } S < S^* \\
  S - K, & \text{when } S \geq S^*
\end{cases}
\]

Where
\[
S^* - K = c(S^*, \tau) + A_2 \\
A_2 = S^* \left[ 1 - e^{-(b-\tau)^T} \left( N(d_1(S^*)) \right) \right] / q_2 \\
q_2 = \frac{1}{2} \left[ -(N-1) + \sqrt{(N-1)^2 + \frac{4M}{H}} \right]
\]

The price of American put option could be derived in the similar way.
\[
P(S, \tau) = \begin{cases} 
  p(S, \tau) + A_1 \left( \frac{S}{S^{**}} \right)^{q_1}, & \text{when } S > S^{**} \\
  K - S, & \text{when } S \leq S^{**}
\end{cases}
\]

Where
\[
K - S^{**} = p(S^{**}, \tau) + A_1 \\
A_1 = -S^{**} \left[ 1 - e^{-(b-\tau)^T} \left( N(-d_1(S^{**})) \right) \right] / q_1 \\
q_1 = \frac{1}{2} \left[ -(N-1) - \sqrt{(N-1)^2 + \frac{4M}{H}} \right]
\]

2.2.2 Numerical Analysis Method

Now the most maturely pricing methods are numerical analysis method, including binary tree method, Finite-difference method and Monte-Carlo method. The famous binary tree method was created in 1979 (Cox, et al., 1979),
it became one of the widely used pricing methods. Its mathematical principle is to use a lot of discrete small range two-value movement to simulate continuous assess price movement, and then use reverse pricing to make price for options. When making price for European option, under some condition, its solution approximate to the solution of B-S model. While pricing American option, one should examine the value at every node, and at the last node the value of American option is equal to that of European option. Its advantage is easy principle and achievement while its disadvantage is low precision (Kind, 2005).

Finite-difference method is also a method to get approximately solution of option price. Its principle is transforming partial differential equation into a series difference equation and then using iterative method to get the price. Finite-difference method divides into dominant and recessive. Its calculation just likes binary tree method; also calculate from the end to beginning. Finite-difference method also fits American pricing. Its disadvantage is not good at processing path-relied option and high dimensional option pricing.

The principle of Monte-Carlo method is to simulate assess path for forecasting the average respond of expected option and getting the estimative value. Generally, people can directly use Monte-Carlo method to simulate without a deep acquaintance in option pricing model. One of Monte-Carlo method’s advantages is that error convergent rate doesn’t rely on dimension. It makes Monte-Carlo method perfectly suitable for the high dimensional option pricing. But some people didn’t think so when Monte-Carlo was young because Monte-Carlo was a forward simulation but American option needed a backward one. Until 1990s Number-Graphic analysis and dynamic planning theory was
introduced into Monte-Carlo method (Li. Y, 2007), achieved American option’s Monte-Carlo simulation pricing step by step. Comparing Monte-Carlo to Binary tree and Finite-difference, its biggest advantage is that it can process multi-underlying asset and path-relied options’ pricing.

3. Monte Carlo Simulation Methods

U.S. scientists Von Neumann and Ulam (1946) invented this method when they do the study of nuclear physics. But the first use of Monte Carlo simulation in the financial field is Boyle (1977). He used this method to price European stock option with single asset (Zheng,X. & Cheng,J., 1999).

The basic principles of Monte Carlo simulation methods are: use random numbers to pick samples for different sample paths. In the risk-neutral world, the underlying assets of the variable follow these tracks. Payoff on expiration date can be calculated easily and then discount payoff to the current time, thus we could get the price of underlying asset. Let the arithmetic average of prices in all sample paths be the estimate of the value of the underlying asset, and then using this estimate to price the derivative securities. This method can be successfully achieved by two important theorems in probability theory: Strong Law of large numbers and Central Limit Theorem. They can ensure that the estimator could converge to the real value of the population.

At present, among the researches of Monte Carlo methods, Least Squares Monte Carlo simulation methods (LSM), proposed by Longstaff and Schwartz (2001), has a great impact. This method has become the standard method of pricing American options (Wu and Xuan, 2006). The numerical experiment in
this paper is fundamentally based on LSM method.

The basic idea of this method is that, in a finite number of discrete point, simulate prices of underlying assets in all sample paths. At each point, compare the profit of exercising option immediately against the expected profit of holding option. Using least-squares method to estimate the profit of holding option, we get the option price in each time point.

In order to simplify the discussion, assume an American-style put option based on a non-dividend stock, using Monte Carlo simulation generated M trading point in time, \( S_0, S_1, ..., S_j, ..., S_M \), where \( S_j = S(j\Delta t) \). The expiration date is \( T = M\Delta t \). \( I_j(S_j) \) is defined as the intrinsic value of options at time \( j \), so at time \( j \), the actual value of options as follows:

\[
V_j(S_j) = \max \left\{ I_j(S_j), E_j \left[ e^{-r\Delta t} V_{j+1}(\hat{S}_{j+1}) | S_j \right] \right\}
\]

(3.1)

Since it is an American-style put option, \( I_j(S_j) = \max \{ K - S_j, 0 \} \). \( r \) is risk free rate. \( E_j \left[ e^{-r\Delta t} V_{j+1}(\hat{S}_{j+1}) | S_j \right] \) is continuation value at time \( j \).

The greatest contribution of Least-squares simulation method is to use a simple polynomial function to approximate the conditional expectation. The approximate equation is presented as follow:

\[
E_j \left[ e^{-r\Delta t} V_{j+1}(\hat{S}_{j+1}) | S_j \right] \approx \sum_{l=1}^{L} \alpha_j S_j^{l-1}
\]

(3.2)

the parameters \( \alpha_j \), can be obtained by linear regression. If we are at time step \( j \) and consider path \( i \), the equation of this regression can be presented as:

\[
e^{-r(j^* - j)\Delta t} \max \{ K - S_{j^* i}, 0 \} = \alpha_{1j} + \alpha_{2j} S_{ji} + \alpha_{3j} S_{ji}^2 + \epsilon_i
\]
Where $j^*$ is exercise time. Since there can be at most one exercise time for each path, it may be the case that after comparing the intrinsic value with the continuation value on a path, the exercise time $j^*$ is reset to a previous period. The question of what kind of polynomial should be chosen to approximate this expectation is still a controversial issue. Some research results show that the higher power of the polynomial the better this approximation will be (Brandimarte, 2006). In this paper, we use $l=3$ which is the same as Longstaff and Schwartz (2001) used in their original papers. Specifically we use $\alpha_{ij} + \alpha_{2j}S_j + \alpha_{3j}S_j^2$ to approximate conditional expectation $E_j$.

4. Quasi Monte-Carlo Simulation

The only difference between Quasi Monte-Carlo and Monte-Carlo is Monte-Carlo using pseudo random numbers to generate the price of underlying asset while Quasi Monte-Carlo using low-discrepancy sequence whose distribution is much more even. It has been shown in the literature that Monte-Carlo’s error convergent rate is $O(n^{-1/2})$, but Quasi Monte-Carlo’s is $O((\ln n)^d/n)$ with low difference sequence (Luo and Xu, 2008). Quasi Monte-Carlo is more attractive when dimensionality is low.

Monte-Carlo simulation and Quasi Monte-Carlo simulation were both used in physics at the beginning, so most solutions were given by physicists. And there had been a controversy about whether these solutions also applicable in finance and derivatives pricing. Corwin Joy (Corwin, 1996) pointed out the disadvantage of Monte-Carlo and raised Quasi Monte-Carlo. They substituted
random sequence with deterministic sequence which can improve convergent rate and have a deterministic error interval. This solution was validated in European option pricing. But there is not so much research focus on Quasi Monte-Carlo when applying to American option pricing, we will compare the efforts of these two methods in American option pricing in Section 5.

4.1. Pseudo Random Numbers

In all simulation, the random numbers generated by computer have a great influence on the result. Most of the advanced programming languages have a function (like “rand”) to generate random numbers from the standard uniform distribution. One can make some changes on it if one wants some other distributions. But actually the random numbers given by computer are not the “real” ones, the computer uses a function named Linear Congruential Generator to generate them, so they also called pseudo random numbers.

Given a non-negative integer $Z_{i-1}$, one can generate a series of random number by the following formula:

$$Z_i = (aZ_{i-1} + c) \mod m$$

where parameter “a” is called multiplier, “c” is called shift and “m” is called modulo. The computer’s word length is often chosen as modulo, $m = 2^{32}$ for example. By using $Z_i / m$ we can get the random numbers in $(0,1)$.

4.2. Low-discrepancy Sequence

The low-discrepancy sequence is much more even in distribution than pseudo random numbers. We use bias rate to describe it, lower bias rate, more even distribution. We will introduce two kinds of widely used low difference
sequence, Halton Sequence and Sobol Sequence.

4.2.1 Halton Sequence

We generate a Halton Sequence with following steps:

1. A decimal integer \( n \), can be expressed as:
   \[
   n = (\cdots d_4 d_3 d_2 d_1 d_0)_b \quad \text{or} \quad n = \sum_{k=0}^{m} d_k b^k
   \]  
   \( (4.1) \)
   Parameter \( m \) is the smallest integer which satisfied \( 0 \leq d_k < b^k \). When \( k > m \) let \( d_k = 0 \). For example, assume we have dimension \( s=1 \) (also known as Van der Corput sequence), \( b=2 \), \( n = 11 = 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 \), so \( n=(1011) \) now.

2. Transform the numbers above to decimal fraction by reflection. That is \( h = (0.d_0 d_1 d_2 d_3 \cdots)_b \). In this example, \( h=(0.1101)_2 \)

3. Express the number based on \( b \) by decimal, it is \( h(n, b) = \sum_{k=0}^{m} d_k b^{-(k+1)} \). In this example 0.1101 is transformed to:
   \[
   (0.1011)_2 = 1 \times 2^{-1} + 1 \times 2^{-2} + 0 \times 2^{-3} + 1 \times 2^{-4} = \frac{13}{16} = 0.8125
   \]

Halton sequences are obtained in multiple dimensions when a Van der Corput generator is associated to each dimension. The figure 4.1 and 4.2 are distributive graphs of pseudo random numbers and Halton sequence in two dimensions. The prime bases we choose are 2 and 7.
4.2.2 Sobol Sequence

The Sobol sequence is a reordering of Halton sequence with base $b=2$ for all dimensions.

1. An arbitrary decimal integer $n$, can be expressed as:

$$n = \sum_{j=0}^{m} a_j 2^j$$

where, $m$ is the least integer larger than or equal to $\log_2 n$. $a_j$ equal to 0 or 1.

2. Given a primitive polynomial of degree $d$,

$$P = x^d + h_1 x^d-1 + h_2 x^d-2 + ... + h_{d-1} x + 1,$$

where, $h_j (j=1,2,...,d-1)$ equal to 0 or 1, we use $h_j$ and the following recurrence formula to generate direction numbers $(v_i)$

$$v_i = h_1 v_{i-1} \oplus h_2 v_{i-2} \oplus ... \oplus h_{d-1} v_{i-d+1} \oplus v_{i-d} \oplus \lfloor v_{i-d} / 2^d \rfloor, \quad i > d$$

where, $\oplus$ is the exclusive-or (XOR) operator. For example
1 \oplus 0 = 0 \oplus 1 = 1, 1 \oplus 1 = 0 \oplus 0 = 0.

3. If we let \( v_i = \frac{m}{2^i} \), where \( m_i < 2^i \) is an odd integer, this is better implemented in integer arithmetic as
\[
m_i = 2h_i m_{i-1} \oplus 2^2 h_2 m_{i-2} \oplus \cdots \oplus 2^{d-1} h_{d-1} m_{i-d+1} \oplus 2^d m_{i-d} \oplus m_{i-d}
\]
Some number \( m_1, \ldots, m_d \) are needed to initialize the recursion.

4. Then the \( n^{th} \) number in sobol sequence generated in the following way
\[
\phi(n) = a_1 v(2) \oplus a_2 v(3) \oplus \cdots \oplus a_i v(i)
\]

5. In order to generate sobol sequence faster, Antonov and Saleev (1979) suggested an improved algorithm.
\[
\phi(n + 1) = \phi(n) \oplus v(i),
\]
Where, \( \phi(0) = 0, \ v(i) \) is the \( i^{th} \) direction number, and \( i \) is the rightmost zero-bit in the base 2 expansion of \( n-1 \).

5. Empirical Study

Consider an American put Option based on non-dividend stock. Under the assumption of Risk-neutral expectation, the stock price follows the Geometric Brownian motion.
\[
dS = rSdt + \sigma SdW
\]
Where \( r \) is Risk-free rate of return, \( \sigma \) is Stock Volatility, \( W \) is a Standard Wiener process. Applying the \( \dot{I}t\hata\)'s lemma, we get,
\[
d \ln S = (r - \frac{\sigma^2}{2}) dt + \sigma dW
\] (5.1)
In order to apply the Monte Carlo simulation, we need to discretize the time interval \([0, T]\) with a time \(\Delta t\) such that \(T = M \cdot \Delta t\). Therefore, (5.1) could be rewritten as follow:

\[
\ln S_j - \ln S_{j-1} = \left( r - \frac{\sigma^2}{2} \right) \Delta t + \sigma \varepsilon_j \sqrt{\Delta t} \quad (5.2)
\]

Where \(\varepsilon_j\) follows standard normal distribution.

Thus if we know the initial value, \(S_0\), then we could get the stock price at time \(j\) which belongs to \([0, T]\), the stock price satisfy the following equation.

\[
S_j = S_0 e^{j \left[ (r - \frac{\sigma^2}{2}) \Delta t + \sigma \varepsilon_j \sqrt{\Delta t} \right]} \quad (5.3)
\]

This equation implies that we could generate a sample path \(h_i(S_0, S_1, ..., S_T)\), where \(i=1,2,..N\), \(N\) is the size of the sample path.

In this study, values of all parameters are set as follows: stock price at issue date \(S_0 = 50\); the exercise price \(K = 50\); risk-free interest rate \(r = 0.05\); the stock volatility \(\sigma = 0.4\); option period (years) \(T = 1\); and \(M=50\) time-steps for potential exercise points, which is straightforward yields \(T = 50 \Delta t\). The size of sample path \(N\) is depending on the accuracy we need. The parameter \(K\) in the equation (3.2) equals to 3. Once all the parameters are determined, the value of American option could be calculated directly. We use R to do all the simulation and calculation.

5.1 Pricing American option use Monte Carlo Simulation.

In the equation (5.3), only the variable \(\varepsilon_i\) is a random variable which follows the standard normal distribution. Thus, we could use the command "rnorm" to generate pseudo random numbers and then substitute them to the equation.
(5.3), as a result we could get the value of stock price in each independent sample path. After we simulate the stock price, we substitute these result to the equation (3.1), then we get the value of American put option. Specifically speaking, the pricing process has been done in these steps:

1. Generate the standard normally distributed random numbers;
2. Substitute the random numbers to the geometric Brownian motion, then generate the sample paths of stock price;
3. Calculate the cash flow, defined by \( \max\{K - S_j, 0\} \), of each sample paths at expiration date \( j=M \).
4. Pick up the sample paths whose profit at expiration date is in-the-money, which means \( K - S_j \) is positive.
5. Discount the cash flow of each path we picked to the previous exercise date \( j=M-1 \);
6. Approximate the conditional expectation function by normal least square method;
7. Apply the result in step 6 to calculate the continuation value;
8. Compare intrinsic and continuation value at time \( j \) and decide which strategy to choose, hold or exercise;
9. Use the result in step 8 as the cash flow at current time;
10. Go to step 4-9 at time \( j=M-1,M-2 \) until \( j=0 \);
11. Calculate the average cash flow of all sample paths at time \( j=0 \).

5.2 Pricing American option use Quasi Monte Carlo Simulation.

Because the random numbers generated by QMC is not normally
distributed, we need to transform them to normal distribution. Generally, there are two options to do the transformation, one is Box-Muller method and the other is inverse transform method. Because of the effect of swapping random numbers in Box-Muller transformation (Brandimarte. P, 2006) which may reduce the performance of fast convergence, we choose inversion method in this case.

The main difference in terms of the experiment steps between Monte Carlo simulation and Quasi Monte Carlo simulation is the first two steps. Specifically speaking, in the case of Quasi Monte Carlo simulation:

1. Generate low-discrepancy sequence in internal $[0,1]$
2. Use inversion method to transform the low-discrepancy sequence to normal distributed Sequence.
3. The same as step 2 in Monte Carlo case, and so on.

Now, considering a fixed time steps for potential exercise point, for example $M=50$, we compare the performance of MC and QMC according to different sample size. Since the price of an option is also a random variable, then we do 10 times replications with sample size $N$ and let the mean value be the simulation option price. Table 1 shows the option prices using different sequences. In order to measure the performance of each sequence, we let the price of option based on $N=100,000,000$ sample size as a “true value”, which denoted by $\theta$ and equals to 6.89874. Then, calculate the MSE between simulated value and “true value” using the follow equation:

$$MSE = \frac{1}{10} \sum_{i=1}^{10} (OP_i - \theta)^2.$$  

Table 2 show the results of MSE.
Table 1 compares the option values between simulation methods and
approximation method; meanwhile in the subsection of simulation methods, we compare Monte Carlo simulation and Quasi Monte simulation.

The results show that the estimated values given by approximation method and MC method with small sample size (eg. N=2500) are not close to real value. However, as the sample size increase, the option values generated by MC method gradually converge to real value and therefore show the advantage of simulation method.

Based on the observation given by MC and QMC, Sobol sequence shows a more satisfied result even if the sample size is small. However, no matter which kind of sequences is applied, the simulation method ensures that the estimated value converges to real value as the sample size increases. The only difference is the speed of the process to convergent. QMC method shows a faster convergence speed than MC method. This feature becomes more visible when we look at the measure of Mean Squared Error (MSE). Table 2 shows the result.

From table 2, the reduction of MSE is remarkable as the sample size increase. When the sample size is 50000, the MSE has already reached a high accuracy requirement. Sobol sequence has the smallest MSE and the MSEs of pseudo and Halton sequence are almost the same.

The computational time of a Monte Carlo Approach is also an important concern for the practioners. Table 3 presents the simulation time using different sequences. Table 3 shows, it becomes more and more time consuming as the sample size increases. Among these three sequences, pseudo-random sequence takes the least time. Due to the complexity of algorithm of low-discrepancy sequence, Halton sequence consumes much more time which
is twice as Sobol sequence.

<table>
<thead>
<tr>
<th>Sample Size</th>
<th>MC</th>
<th>QMC</th>
</tr>
</thead>
<tbody>
<tr>
<td>N=2500</td>
<td>0.86</td>
<td>1.64</td>
</tr>
<tr>
<td>N=5000</td>
<td>1.56</td>
<td>3.18</td>
</tr>
<tr>
<td>N=10000</td>
<td>3.03</td>
<td>6.4</td>
</tr>
<tr>
<td>N=20000</td>
<td>6.36</td>
<td>12.84</td>
</tr>
<tr>
<td>N=50000</td>
<td>16.26</td>
<td>34.26</td>
</tr>
</tbody>
</table>

Table 3 Simulation time (in Seconds)

After comparing the performance of MC and QMC with fixed time-step for exercise points, we need to figure out how they performed with fixed sample size, set at N=10000.

In this case, we do not have a “true value” of option price, since the sample size is fix. It also does not make sense if we enlarge the time-steps to get a “true value”, because different time-steps means different option contract and therefore the different option price. Table 4 shows the option prices of different time-steps for exercise using different sequences. As discuss in previous section, when the time-steps equals to one, the price of American option should be equal to European option, otherwise, American one is larger than European one. Also, when the numbers of time-steps are getting large, the option price gets higher. Because, more time-steps indicate more opportunity to make the choice, hence one should pay for extra opportunity. Both pseudo-random
sequence and Low-discrepancy sequence show this character of American option.

Table 4  Option price based on fixed sample size (N=10000)

<table>
<thead>
<tr>
<th>Time-steps for exercise</th>
<th>MC</th>
<th>QMC</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pseudo</td>
<td>Halton</td>
</tr>
<tr>
<td>M=8</td>
<td>5.70</td>
<td>5.77</td>
</tr>
<tr>
<td>M=16</td>
<td>6.36</td>
<td>6.46</td>
</tr>
<tr>
<td>M=32</td>
<td>6.82</td>
<td>6.88</td>
</tr>
<tr>
<td>M=50</td>
<td>6.90</td>
<td>6.86</td>
</tr>
<tr>
<td>M=100</td>
<td>6.93</td>
<td>6.88</td>
</tr>
</tbody>
</table>

Finally, we compare the simulation time in this case. Table 5 reports the results using different sequences. The conclusion is pretty much the same as the one based on fixed time-steps; pseudo-random sequence takes the least time.

Table 5 Simulation time based on N=10000 (in seconds)

<table>
<thead>
<tr>
<th>Time-steps for exercise</th>
<th>MC</th>
<th>QMC</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pseudo</td>
<td>Halton</td>
</tr>
<tr>
<td>M=8</td>
<td>0.63</td>
<td>1.12</td>
</tr>
<tr>
<td>M=16</td>
<td>1.14</td>
<td>4.18</td>
</tr>
<tr>
<td>M=32</td>
<td>1.92</td>
<td>4.28</td>
</tr>
<tr>
<td>M=50</td>
<td>3.62</td>
<td>6.4</td>
</tr>
<tr>
<td>M=100</td>
<td>6.22</td>
<td>13.69</td>
</tr>
</tbody>
</table>
6 Conclusion

This paper briefly summarizes some popular American-style option pricing methods. Based on the least squares Monte Carlo algorithm, we compare the performance of Monte Carlo simulation and Quasi Monte Carlo simulation also compare the simulation method and quadratic approximation method for a put option with a single underlying asset. The results show that quasi-Monte Carlo with Sobol sequence perform better in terms of fast convergence and stabilization than standard Monte Carlo Method. However, all what we did is based on the simulation experiment under an extremely theoretic environment. If we want to test whether above conclusions are also valid in real situation, we could use a real option instead of an a simulated one and apply the model to get the estimated value and then to compare with the real one. Since the real world is much more complicated than the theoretical one, how to find a proper way to apply Monte Carlo and Quasi-Monte Carlo to a real situation can be a topic of future research.
Reference


